线性代数应该这样学

*[Linear Algebra Done Right]*

1. 向量空间的定义、基本性质
2. 线性相关、张成、基、维数的定义

有限维向量空间的基本理论

1. 引入线性映射（主要结果：线性映射的零空间的维数加上值域的维数等于定义域的维数）
2. 多项式部分理论——理解线性算子所必需
3. 引入本征向量——将线性算子限制到更小到子空间上来研究

复向量空间上本征值存在性到简洁证明

复向量空间上的线性算子关于某个基有上三角矩阵

实向量空间上的线性算子都具有1维或2维的不变子空间

奇数维实向量空间上的线性算子都有本征值

1. 内积空间的定义、基本性质

标准工具（规范正交基、格拉姆-施密特正交化过程以及伴随......）

利用正交投影解极小化问题

1. 谱定理（刻画了本征向量可以组成规范正交基的线性算子）

正定算子、线性等距同构、极分解以及奇异值分解

1. 引入极小多项式、特征多项式以及广义本征向量

（主要成果：用广义本征向量来描述复向量空间上的线性算子，可证几乎所有通常要使用Jordan形来证明的结果）

（复向量空间上的可逆线性算子都有平方根，复向量空间上的线性算子都有Jordan形......）

1. 实向量空间上的线性算子（核心）

2维不变子空间弥补了此类算子可能无本征值的不足，得到与复向量空间类似的结果

1. 利用特征多项式给出迹和行列式的定义（前面定义特征多项式时并未使用行列式）

复向量空间上，定义的另一种陈述方式：迹是所有本征值之和；行列式是所有本征值之积（两种情况都计重数）

利用极分解和自伴算子的刻画导出多重积分的换元公式

1. 向量空间
2. 复数
3. 向量空间的定义
4. 向量空间的定义
5. 子空间
6. 和与直和
7. 习题
8. 有限维向量空间
9. 张成与线性无关
10. 基
11. 维数
12. 习题
13. 线性映射
14. 定义与例子
15. 零空间与值域
16. 线性映射的矩阵
17. 可逆性
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28. 对角矩阵
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30. 习题
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32. 内积
33. 范数
34. 规范正交基
35. 正交投影与极小化问题
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47. 广义本征向量
48. 特征多项式
49. 算子的分解
50. 平方根
51. 极小多项式
52. 约当形
53. 习题
54. 实向量空间上的算子
55. 方阵的本征值
56. 分块上三角矩阵
57. 特征多项式
58. 习题
59. 迹与行列式
60. 基变换
61. 迹
62. 算子的行列式
63. 矩阵的行列式
64. 体积

符号索引

索引

1.0向量空间

1.1复数(complex number)

1. $i=\sqrt{-1}$
2. $i^{2}=-1$
3. 复数集C：$C=\left\{a+bi：a,b\in R\right\}$
4. 实数集R
5. R(表示C或R)中的元素称——标量(scalar)

1.2向量空间的定义

* $F^{n}=\left\{\left(x\_{1},…,x\_{n}\right):x\_{i}\in F, i=1,…,n\right\}$
* Fn中的元素$x=\left(x\_{1},…,x\_{n}\right)$看作一个始于原点的箭头——向量(vector)
* Fn上的加法：通过相应坐标相加定义两个元素的和
* Fn上的乘法：F中元素与Fn中元素的乘法——标量乘法
* 向量空间(vector space)：带有加法和标量乘法的集合V
* V上的加法(addition)：指一个函数，把每一对每一对$u,v\in V$，都对应到V的一个元素$u+v$
* V上的标量乘法(scalar multiplication)：指一个函数，把任意$a\in F,v\in V$，对应到V的一个元素$uv\in V$

Linear Algbra Done Right

*Preface for the Instructor*

*Preface for the Student*

*1. Vector Space*

*1.1 Rn and Cn*

*1.1.1 Complex Numbers*

*1.1.2 Lists*

*1.1.3 Fn*

*1.1.4 Digression on Fields*

*1.1.n Exercises*

*1.2 Definition of Vector Space*

*1.2.n Exercises*

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*1.3.2 Direct Sums*

*1.3.n Exercises*

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*2.1.1 Linear Combinations and Span*

*2.1.2 Linear Independence*

*2.1.n Exercises*

*2.2 Bases*

*2.2.n Exercises*

*2.3 Dimension*

*2.3.n Exercises*

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*3.3.3 Matrix Multiplication*

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*3.4.3 Linear Maps Thought of as Matrix Multiplication*

*3.4.4 Operators*

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*3.5 Products and Quotients of Vector Spaces*

*3.5.1 Products of Vector Spaces*

*3.5.2 Products and Direct Sums*

*3.5.3 Quotients of Vector Spaces*

*3.5.n Exercises*

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*3.6.4 The Rank of a Matrix*

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*4.0.2 Uniqueness of Coefficients for Polynomials*

*4.0.3 The Division Algorithm for Polynomials*

*4.0.4 Zeros of Polynomials*

*4.0.5 Factorization of Polynomials over C*

*4.0.6 Factorization of Polynomials over R*

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*5.1.2 Restriction and Quotient Operators*

*5.1.n Exercises*

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*5.2.1 Polynomials Applied to Operators*

*5.2.2 Existence of Eigenvalues*

*5.2.3 Upper-Triangular Matrices*

*5.2.n Exercises*

*5.3 Eigenspaces and Diagonal Matrices*

*5.3.n Exercises*

*6. Inner Product Spaces*

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*6.1.1 Inner Products*

*6.1.2 Norms*

*6.1.n Exercises*

*6.2 Orthonormal Bases*

*6.2.1 Linear Functionals on Inner Product Spaces*

*6.2.n Exercises*

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*10.2.2 Determinant of a Matrix*

*10.2.3 The Sign of the Determinant*

*10.2.n Exercises*

**Chapter 1 Vector Space**

Vector spaces are a generalization of the description of a plane using two coordinates, as published by *Descarts* in 1637.

*1. Vector Space*

Linear algebra is the study of linear maps on finite-dimensional vector spaces.

Learning properties of the complex numbers

* basic properties of the complex numbers
* Rn and Cn
* vector spaces
* subspaces
* sums and direct sums of subspaces

*1.1 Rn and Cn*

1.1.1 Complex Numbers

R: the set of real numbers

i: denote a square root of -1, that obeys the usual rules of arithmetic

1.1.2 Lists

1.1.3 Fn

1.1.4 Digression on Fields

1.1.n Exercises

1.2 Definition of Vector Space

1.2.n Exercises

1.3 Subspaces

1.3.1 Sums of Subspaces

1.3.2 Direct Sums

1.3.n Exercises

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1.2 Lengths and Dot Products

1.3 Matrices

2. Solving Linear Equations

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2.2 The Idea of Elimination

2.3 Elimination Using Matrices

2.4 Rules for Matrix Operations

2.5 Inverse Matrices

2.6 Elimination = Factorization: A = LU

2.7 Transposes and Permutations

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3.3 The Complete Solution to Ax = b

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4.3 Least Subquares Approximations

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8.2 The Matrix of a Linear Transformation

8.3 The Search for a Good Basis

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9.2 Hermitian and Unitary Matrices

9.3 The Fast Fourier Transform

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10.2 Matrices in Engineering

10.3 Markov Matrices, Population and Economices

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Lecture 3: Multiplication and inverse matrices

Lecture 4: Factorization into A = LU

Lecture 5: Transposes, permutations, spaces R^n

Lecture 6: Column space and nullspace

Lecture 7: Solving Ax = 0: pivot variables, special solutions

Lecture 8: Solving Ax = b: row reduced form R

Lecture 9: Independence, basis, and dimension

Lecture 10: The four fundamental subspaces

Lecture 11: Matrix spaces; rank 1; small world graphs

Lecture 12: Graphs, networks, incidence matrices

Lecture 13: Quiz 1 review

Lecture 14: Orthogonal vectors and subspaces

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**1. The geometry of linear equations**

n linear equations & n unknowns → row picture & \*column picture → matrix form, $AX=b$

$$\left\{\begin{matrix}a\_{11}x+a\_{12}y=b\_{1}\\a\_{21}x+a\_{22}y=b\_{2}\end{matrix}\right. \leftrightarrow \left[\begin{matrix}a\_{11}&a\_{12}\\a\_{21}&a\_{22}\end{matrix}\right]\left[\begin{matrix}x\\y\end{matrix}\right]=\left[\begin{matrix}b\_{1}\\b\_{2}\end{matrix}\right] \leftrightarrow x\left[\begin{matrix}a\_{11}\\a\_{21}\end{matrix}\right]+·y\left[\begin{matrix}a\_{12}\\a\_{22}\end{matrix}\right]=\left[\begin{matrix}b\_{1}\\b\_{2}\end{matrix}\right]$$

$$\left\{\begin{matrix}a\_{11}x+a\_{12}y+a\_{13}z=b\_{1}\\a\_{21}x+a\_{22}y+a\_{23}z=b\_{2}\\a\_{31}x+a\_{32}y+a\_{33}z=b\_{3}\end{matrix}\right. \leftrightarrow \left[\begin{matrix}a\_{11}&a\_{12}&a\_{13}\\a\_{21}&a\_{22}&a\_{23}\\a\_{31}&a\_{32}&a\_{33}\end{matrix}\right]\left[\begin{matrix}x\\y\\z\end{matrix}\right]=\left[\begin{matrix}b\_{1}\\b\_{2}\\b\_{3}\end{matrix}\right] \leftrightarrow x\left[\begin{matrix}a\_{11}\\a\_{21}\\a\_{31}\end{matrix}\right]+y\left[\begin{matrix}a\_{12}\\a\_{22}\\a\_{32}\end{matrix}\right]+z\left[\begin{matrix}a\_{13}\\a\_{23}\\a\_{33}\end{matrix}\right]=\left[\begin{matrix}b\_{1}\\b\_{2}\\b\_{3}\end{matrix}\right]$$

**2. Elimination with matrices**

elimination → success/failure → back-substitution, elimination matrices, matrix multiplication → A to U(upper triangular)

$$\left[\begin{matrix}a\_{11}&a\_{12}&a\_{13}\\a\_{21}&a\_{22}&a\_{23}\\a\_{31}&a\_{32}&a\_{33}\end{matrix}\right]\rightarrow \left[\begin{matrix}a\_{11}&a\_{12}&a\_{13}\\0&b\_{22}&b\_{23}\\0&b\_{32}&b\_{33}\end{matrix}\right]\rightarrow \left[\begin{matrix}a\_{11}&a\_{12}&a\_{13}\\0&b\_{22}&b\_{23}\\0&0&c\_{33}\end{matrix}\right] 3 pivots(a\_{11},b\_{22},c\_{33}\ne 0)$$

**3. Multiplication and inverse matrices**

**4. Factorization into A = LU**

**5. Transposes, permutations, spaces R^n**

**6. Column space and nullspace**

**7. Solving Ax = 0: pivot variables, special solutions**

**8. Solving Ax = b: row reduced form R**

**9. Independence, basis, and dimension**

**10. The four fundamental subspaces**

Elementary Number Theory and Its Applications

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